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Extended Kac–Ahiezer Formula for the Fredholm Determinant of Integral Operators

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1. INTRODUCTION

It is a familiar result that if T is a Hilbert–Schmidt operator on a Hilbert space with nonzero eigenvalues $\lambda_1, \lambda_2, \ldots$ repeated according to multiplicity, then the infinite product

$$\varphi_\lambda(T) = \prod_{i=1}^{\infty} (1 - \lambda\lambda_i) \exp(\lambda\lambda_i) \tag{1.1}$$

converges and defines an entire function of λ [2, p. 1036]. For special operators, other representations for $\varphi_\lambda(T)$ have been obtained (cf. [1, 3, 4]). An extension of one such result is given by Rao [7] to a class of self-adjoint integral operators in $L^2[0, \tau]$ of the form

$$T_\tau f(t) = \int_0^\tau k(t - s) f(s)\, ds. \tag{1.2}$$

The purpose of this paper is to show that Rao's results extend, with essentially no change in proof, to a class of self-adjoint integral operators of the form

$$T_\tau f(t, x) = \int_0^\tau \int_D k(t - s, x, y) f(s, y)\, d\mu(y)\, ds, \tag{1.3}$$

where (D, Σ, μ) is a measure space for D a compact topological space and μ a positive, finite, regular measure. Such operators arise in applications. An example from radiative transfer theory can be found in [5], where D is a finite set and T_τ is a matrix of integral operators with weakly polar kernels.

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2. Resolvent Kernels

We make the following assumptions on the kernel k:

(i) It is real-valued and in $L_2([-a, a] \times D \times D)$ for every $a, 0 \leqslant a < \infty$ and

$$\int_0^\tau \int_D k^2(t - s, x, y)\, d\mu(y)\, ds \leqslant C(\tau) < \infty.$$

(ii) The iterated kernel

$$k_2(t, s, x, y) = \int_0^\tau \int_D k(t - \sigma, x, z)\, k(\sigma - s, z, y)\, d\mu(z)\, d\sigma$$

$$\text{is in } C([0, \tau]^2 \times D^2) \qquad \text{for every } \tau,\ 0 \leqslant \tau < \infty. \tag{2.1}$$

(iii) Almost everywhere in $R \times D \times D$

$$k(t, x, y) = k(-t, x, y) = k(t, y, x).$$

It follows readily from these assumptions that T_τ in (1.3) is defined in $L_2([0, \tau] \times D)$ as a self-adjoint Hilbert–Schmidt integral operator. There is a sequence of eigenvalues $\{\lambda_i(\tau)\}$ with corresponding orthonormal eigenfunctions $\{\varphi_i(t, x; \tau\}$ which are continuous in t and x (by assumption (ii)) and real-valued (by assumptions (i) and (iii)).

If λ is not a characteristic value of T_τ, then the resolvent operator $\lambda(I - \lambda T_\tau)^{-1} T_\tau$ is an integral operator with kernel r. It follows from assumption (i) that the Mercer theorem applies to k_2 and that $(I - \lambda T_\tau)^{-1}$ is a bounded operator in $L_\infty[(0, \tau) \times D]$. Hence

$$r(t, s, x, y; \lambda, \tau) = \lambda k(t - s, x, y) + \lambda^2 \sum_{i=1}^\infty \frac{\lambda_i^2(\tau)}{1 - \lambda\lambda_i(\tau)} \varphi_i(t, x; \tau)\, \varphi_i(s, y; \tau), \tag{2.2}$$

where convergence of the series is in the uniform norm.

We define two Hilbert–Schmidt integral operators, R_1 and R_2, in $L_2(D)$ by the continuous kernels

$$r_1(x, y; \lambda, \tau) = \lim_{\substack{t \to 0 \\ s \to 0}} [r(t, s, x, y; \lambda, \tau) - \lambda k(t - s, x, y)] \tag{2.3}$$

and

$$r_2(x, y; \lambda, \tau) = \lim_{\substack{t \to \tau \\ s - 0}} [r(t, s, x, y; \lambda, \tau) - \lambda k(t - s, x, y)] \tag{2.4}$$

These are trace class operators, because for any orthonormal basis $\{\psi_j\}$ of $L_2(D)$ there is a function M of λ such that

$$\sum_{j=1}^{\infty} |\langle R_1\psi_j, \psi_j\rangle|$$

$$\leqslant M(\lambda) \sum_{i,j=1}^{\infty} \left| \int_0^\tau \int_{D\times D} k(s, x, y)\, \overline{\psi_j(x)}\, \varphi_i(s, y)\, d\mu(x)\, d\mu(y)\, ds \right|^2$$

$$\leqslant M(\lambda) \int_0^\tau \int_{D\times D} k^2(s, x, y)\, d\mu(x)\, d\mu(y)\, ds < \infty,$$

by Bessel's inequality and assumption (i) for λ not a characteristic value. A similar argument for R_2 requires first the observation that, by assumption (iii), an orthonormal set of eigenfunctions $\{\varphi_i\}$ can be chosen so that

$$\varphi_i(\tau - t, x; \tau) = \pm\varphi_i(t, x; \tau). \tag{2.5}$$

We assume this done and write the set of eigenvalues as

$$\{\lambda_1, \lambda_2, \ldots\} = \{\lambda_1^+, \lambda_2^+, \ldots\} \cup \{\lambda_1^-, \lambda_2^-, \ldots\}, \tag{2.6}$$

where λ_i^+ and λ_i^- correspond to eigenfunctions with respective sign in (2.5).

Our main result is

THEOREM 1. *If k satisfies conditions in* (2.1), *then*

$$\prod_{i=1}^{\infty} (1 - \lambda\lambda_i(\tau))\, \mathrm{Exp}(\lambda\lambda_i(\tau)) = \mathrm{Exp}\left[-\int_0^\tau \int_D r_1(x, x; \lambda, \sigma)\, d\mu(x)\, d\sigma\right] \tag{2.7}$$

and

$$\frac{\prod (1 - \lambda\lambda_i^+(\tau))\, \mathrm{Exp}(\lambda\lambda_i^+(\tau))}{\prod (1 - \lambda\lambda_i^-(\tau)\, \mathrm{Exp}(\lambda\lambda_i^-(\tau))} = \mathrm{Exp}\left[-\int_0^\tau \int_D r_2(y, y; \lambda, \sigma)\, d\mu(y)\, d\sigma\right], \tag{2.8}$$

for λ not in the set of characteristic values of T_τ.

This theorem is an immediate consequence of (2.2)–(2.5), the formulas

$$\int_D r_1(x, x; \lambda, \tau)\, d\mu(x) = \lambda^2 \sum_{i=1}^{\infty} \frac{\lambda_i^2(\tau)}{1 - \lambda\lambda_i(\tau)} \int_D \varphi_i^2(\tau, x)\, d\mu(x), \tag{2.9}$$

and

$$\int_D r_2(x, x; \lambda, \tau)\, d\mu(x) = \lambda^2 \sum_{i=1}^{\infty} \frac{\pm(\lambda_i^\pm(\tau))^2}{1 - \lambda\lambda_i^\pm(\tau)} \int_D \varphi_i^2(\tau, x)\, d\mu(x), \tag{2.10}$$

and

THEOREM 2. *If k satisfies conditions in* (2.1), *then each $\lambda_i(\tau)$ is an absolutely continuous function of τ and*

$$\frac{d}{d\tau}\lambda_i(\tau) = \lambda_i(\tau)\int_D \varphi_i^2(\tau, x; \tau)\, d\mu(x). \tag{2.11}$$

Proof. We give only a brief sketch of the detailed proof in [6]. Monotonicity with τ of $\lambda_i(\tau)$ follows from the min–max characterization of eigenvalues. Hence λ_i is differentiable almost everywhere, and the argument in [6] extends to show λ_i absolutely continuous.

The representation (2.11) is proven as in [6] by considering a sequence $\{\tau_j\}$ decreasing to τ, extending $\varphi_i(\tau_j)$, which is normalized in $L_2([0, \tau_j] \times D)$, from $[0, \tau_j] \times D$ to $[0, \tau_1] \times D$ by the integral equation

$$\lambda_i^2(\tau_j)\varphi_i(t, x; \tau_j) = \int_0^{\tau_j}\int_D k_2(t, s, x, y)\, \varphi_i(s, y; \tau_j)\, d\mu(x)\, ds,$$

showing the sequence $\{\varphi_i(\tau_j)\}_{j=1}^{\infty}$ to be uniformly bounded and equicontinuous on $[0, \tau_1] \times D$, and obtaining a subsequence for which

$$\begin{aligned}\frac{\lambda_i(\tau_{j'}) - \lambda_i(\tau)}{\tau_{j'} - \tau} &\int_0^{\tau}\int_D \varphi_i(t, x; \tau_{j'})\, \varphi_i(t, x; \tau)\, d\mu(x)\, dt \\ &= \frac{\lambda_i(\tau)}{\tau_{j'} - \tau}\int_{\tau}^{\tau_{j'}}\int_D \varphi_i(t, x, \tau_{j'})\, \varphi_i(t, x; \tau)\, d\mu(x)\, dt.\end{aligned}$$

3. KAC–AHIEZER FORMULA

The results in Theorem 1 are interesting when the kernels r_1 and r_2 can be written as

$$r_1(x, y; \lambda, \tau) = r(0, 0, x, y; \lambda, \tau) - \lambda k(0, x, y) \tag{3.1}$$

and

$$r_2(x, y; \lambda, \tau) = r(\tau, 0, x, y; \lambda, \tau) - \lambda k(\tau, x, y) \tag{3.2}$$

for almost all τ. With this in mind we define

Condition I. For some $\epsilon > 0$, k is continuous in $[-\epsilon, \epsilon] \times D \times D$.

Condition II. The function k defines a mapping from the real line to the Banach space $C(D \times D)$ which is continuous almost everywhere and Bochner integrable.

We consider the integral equation

$$\alpha(t, x, y; \lambda, \tau) = \lambda k(t, x, y) + \lambda \int_0^\tau \int_D k(t-s, x, z)\, \alpha(x, z, y; \lambda, \tau)\, d\mu(z)\, ds. \tag{3.3}$$

If Condition I holds, then

$$r_1(x, y; \lambda, \tau) = \alpha(0, x, y; \lambda, \tau) - \lambda k(0, x, y),$$

and operators K_1 and A_1 are defined in $L_2(D)$ by continuous kernels $k(0, x, y)$ and $\alpha(0, x, y; \lambda, \tau)$ so that

$$R_1 = A_1 - \lambda K_1 .$$

If Condition II holds, then

$$r_2(x, y; \lambda, \tau) = \alpha(\tau, x, y; \lambda, \tau) - \lambda k(\tau, x, y)$$

for almost all τ.

We restate Theorem 1.

COROLLARY 1. *If k satisfies* (2.1) *and Condition* I, *then for λ satisfying $\lambda\lambda_i \neq 1$ for $i = 1, 2,\ldots$,*

$$\prod_{i=1}^{\infty} (1 - \lambda\lambda_i(\tau))\, \mathrm{Exp}(\lambda\lambda_i(\tau)) = \mathrm{Exp}\left[\lambda\tau \int_D k(0, x, x)\, d\mu(x) - \int_0^\tau \int_D \alpha(0, x, x; \lambda, \sigma)\, d\mu(x)\, d\sigma\right]. \tag{3.4}$$

COROLLARY 2. *If k satisfies* (2.1) *and Condition* II, *then for λ satisfying $\lambda\lambda_i \neq 1$ for $i = 1, 2,\ldots$,*

$$\frac{\prod (1 - \lambda\lambda_+{}^i(\tau))\, \mathrm{Exp}(\lambda\lambda_i{}^+(\tau))}{\prod (1 - \lambda\lambda_i{}^-(\tau))\, \mathrm{Exp}(\lambda\lambda_i{}^-(\tau))} = \mathrm{Exp}\left[\int_0^\tau \int_D [\lambda k(\sigma, x, x) - \alpha(\sigma, x, x; \lambda, \sigma)]\, d\mu(x)\, d\sigma\right].$$

Any additional smoothness condition on k which guarantees uniform convergence of the representation

$$k(t-s, x, y) = \sum_{i=1}^{\infty} \lambda_i \varphi_i(t, x)\, \varphi_i(s, y)$$

gives, by Theorem 2, the Kac–Ahiezer formula

$$\prod_{i=1}^{\infty} (1 - \lambda\lambda_i(\tau)) = \operatorname{Exp} \left[-\int_0^\tau \int_D \alpha(0, x, x; \lambda, \sigma) \, d\mu(x) \, d\sigma \right]. \tag{3.6}$$

The following is sufficient for this to follow from the Mercer theorem:

Condition III. The real-valued kernel k satisfies (iii) of (2.1), is in $C[(-\infty, \infty) \times D^2] \cap L_2[(-\infty, \infty) \times D^2]$, and the Fourier transform in t,

$$\hat{k}(\xi, x, y) = \int_{-\infty}^{\infty} k(t, x, y) \operatorname{Exp}(it\xi) \, dt,$$

defines a positive Hermitian operator in $L_2(D)$ for every ξ, $-\infty < \xi < \infty$.

For real λ, not a characteristic value of T_τ, R_1 and R_2 are self-adjoint operators in $L_2(D)$ and R_1 is quasi-definite for each such λ. By the Mercer theorem we can compute the trace of $R_1(\lambda, \tau)$ by

$$\operatorname{Tr} R_1(\lambda, \tau) = \int_D r_1(x, x; \lambda, \tau) \, d\mu(x).$$

We have

COROLLARY 3. *If k satisfies* (2.1) *and Condition* I, *and if K_1 is quasi-definite, then for real λ satisfying $\lambda\lambda_i \neq 1$ for $i = 1, 2,...$*

$$\prod_{i=1}^{\infty} (1 - \lambda\lambda_i(\tau)) \operatorname{Exp}(\lambda\lambda_i(\tau))$$

$$= \operatorname{Exp} \left[\lambda\tau \operatorname{Tr} K_1 - \int_0^\tau \operatorname{Tr} A_1(\lambda, \sigma) \, d\sigma \right]. \tag{3.7}$$

If k satisfies Condition III, *then*

$$\prod_{i=1}^{\infty} (1 - \lambda\lambda_i(\tau)) = \operatorname{Exp} \left[-\int_0^\tau \operatorname{Tr} A_1(\lambda, \sigma) \, d\sigma \right]. \tag{3.8}$$

In special cases it is also possible to express (3.5) in terms of the traces of operators. In particular, for D a finite set, (3.4)–(3.6) can be expressed in terms of traces of matrices for all λ outside the set of characteristic values of T_τ.

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